ON A SOLUTION TO THE EQUATIONS OF MAGNETO-GASDYNAMICS

(K RESHENIIU URAVNENII MAGNITNOI GAZODINAMIKI)

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I.M. IUR'EV
(Moscow)

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An investigation of strong and weak discontinuities in magneto-hydrodynamics is contained in a series of papers and books (see, for instance, [1-4]). In the following, the equations of planar flow in a magnetic field parallel to the velocity field are transformed under certain initial restraints to a linear equation of the Chaplygin type [5]. We will apply the result to a problem in which there are no strong discontinuities.

The equations of the steady motion of a gas with infinite conductivity in a magnetic field have the following form:

$$\operatorname{div} \mathbf{H}=0, \quad \operatorname{rot}(\mathbf{W}\times\mathbf{H})=0, \quad \operatorname{div}\rho\mathbf{W}=0, \quad (\mathbf{W}\cdot\nabla)\mathbf{W}=-\frac{\operatorname{grad} p}{\rho}-\frac{1}{4\pi\rho}\mathbf{H}\times\operatorname{rot}\mathbf{H} \qquad (1)$$

where $\mathbf{H}$ is the magnetic field strength, p, ρ and $\mathbf{W}$ are respectively the pressure, density and vector velocity of the flow. If the flow is planar and the vector $\mathbf{H}$ lies in the plane of the flow, it follows from the second of Equations (1) that $\mathbf{W}\times\mathbf{H} = \text{const}$. If $\mathbf{W} \parallel \mathbf{H}$ at one point, then $\mathbf{W} \parallel \mathbf{H}$ throughout the flow field. One can write

$$\mathbf{H} = k(x, y)\,\rho\mathbf{W} \qquad (2)$$

where $k(x, y)$ is the coefficient of proportionality.

From the first and third equations of the system (1) we conclude that $k(x, y) = \text{const}$ along a streamline. The vector $\mathbf{H}\times\operatorname{rot}\mathbf{H}$ is perpendicular to the streamline. Therefore, the Bernoulli formula

$$w\,dw+\frac{dp}{\rho}=0 \qquad (3)$$

is correct along streamlines.

Let us assume $p = p(\rho)$ and let formula (3) be correct in any direction in the region of flow. We will also consider subsequently that $k = \text{const}$ throughout the flow, which obtains in particular for the undisturbed parallel flow at infinity. On the basis of (3) we have

$$\rho = \rho(w), \qquad p = -\int \rho(w)\, w\, dw \tag{4}$$

from which there follows

$$\frac{1}{\rho}\operatorname{grad} p = -\operatorname{grad}\frac{w^2}{2} \tag{5}$$

On the other hand, $(\mathbf{W}\cdot\nabla)\,\mathbf{W} = \operatorname{rot}\mathbf{W}\times\mathbf{W} + \operatorname{grad}{}^1/_2\,\mathbf{W}^2$. Therefore, the last equation of the system (1) reduces to the form

$$\operatorname{rot}\mathbf{W}\times\mathbf{W} = -\frac{1}{4\pi\rho}\mathbf{H}\times\operatorname{rot}\mathbf{H} \tag{6}$$

Projecting Equation (6) on to the coordinate axes x, y and taking formula (2) into account, we obtain

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{k^2}{4\pi}\left(\frac{\partial \rho v}{\partial x} - \frac{\partial \rho u}{\partial y}\right), \quad \text{или} \quad \frac{\partial v^*}{\partial x} - \frac{\partial u^*}{\partial y} = 0 \tag{7}$$

where

$$u^* = w^*\cos\theta, \qquad v^* = w^*\sin\theta, \qquad w^* = w\left(1 - \frac{k^2}{4\pi}\rho\right) \tag{8}$$

and θ is the angle of the velocity vector with the abscissa.

The existence of a stream function $\psi(x, y)$

$$\frac{\partial\psi}{\partial x} = -v\rho(w) = -v^*\rho^*(w^*), \qquad \frac{\partial\psi}{\partial y} = u\rho(w) = u^*\rho^*(w^*) \qquad \left(\rho^* = \frac{\rho}{1 - k^2\rho/4\pi}\right) \tag{9}$$

follows from the continuity equation.

Equation (7) permits a fictitious potential ϕ to be introduced in accordance with the formula

$$\frac{\partial\varphi}{\partial x} = u^*, \quad \frac{\partial\varphi}{\partial y} = v^* \tag{10}$$

As is well-known from the equations of total differential expressions

$$dx = \frac{\cos\theta}{w^*}d\varphi - \frac{\sin\theta}{\rho^* w^*}d\psi, \qquad dy = \frac{\sin\theta}{w^*}d\varphi + \frac{\cos\theta}{\rho^* w^*}d\psi \tag{11}$$

one can derive the following system of equations for the unknown function ϕ and ψ:

$$\frac{\partial\varphi}{\partial\theta} = \frac{w^*}{\rho^*}\frac{\partial\psi}{\partial w^*}, \quad \frac{\partial\varphi}{\partial w^*} = w^*\frac{d}{dw^*}\left(\frac{1}{\rho^* w^*}\right)\frac{\partial\psi}{\partial\theta} \tag{12}$$

The system (12) has the canonical form

$$\frac{\partial\varphi}{\partial\theta} = \sqrt{K}\frac{\partial\psi}{\partial s}, \qquad \frac{\partial\varphi}{\partial s} = -\sqrt{K}\frac{\partial\psi}{\partial\theta} \tag{13}$$

where the functions of the velocity $\sqrt{K}$ and s are related to w^* *and* ρ^* *in* accordance with (6) by the formulas

$$\frac{dQ}{ds} = -\sqrt{K}P, \qquad Q = \sqrt{K}\frac{dP}{ds} \qquad \left(P = w^{*-1},\ Q = (\rho^* w^*)^{-1}\right) \tag{14}$$

hence

$$ds = \left(\frac{P'(w^*)\,Q'(w^*)}{P(w^*)\,Q(w^*)}\right)^{1/2} dw^*, \qquad \sqrt{K} = \left(\frac{Q(w^*)\,Q'(w^*)}{P(w^*)\,P'(w^*)}\right)^{1/2} \tag{15}$$

Substituting expressions for the functions $P(w^*)$ and $Q(w^*)$ into (15) and taking into consideration formulas (8) and (9) and the formula for determining the velocity of sound

$$a^2 = \frac{dp}{d\rho} = -\frac{w\rho(w)}{\rho'(w)} \tag{16}$$

we obtain

$$\sqrt{K} = \frac{1}{\rho}\left(\frac{(1-M^2)(1-m\rho)^3}{1-m\rho(1-M^2)}\right)^{1/2}, \quad ds = \pm\left(\frac{(1-M^2)[1-m\rho(1-M^2)]}{1-m\rho}\right)^{1/2}\frac{dw}{w} \quad (m = k^2/4\pi)$$

where M is the Mach number. The negative sign in (17) is taken for the interval of variation of w in which $dw^*/ds < 0$. For imaginary values of s and $\sqrt{K}$ we have the hyperbolic system of equations

$$\frac{\partial\varphi}{\partial\theta} = \sqrt{\chi}\,\frac{\partial\psi}{\partial\sigma}, \qquad \frac{\partial\varphi}{\partial\sigma} = -\sqrt{\chi}\,\frac{\partial\psi}{\partial\theta} \qquad (\sigma = -is,\ \sqrt{\chi} = -i\sqrt{K}) \tag{18}$$

In the case $p = \text{const}\,\rho^{\kappa}$, where κ is the ratio of specific heat coefficients, formulas (17) and (21) take the form

$$\sqrt{K} = \left(\frac{(1-\lambda^2)[1-k_1(1-\lambda^2/h^2)^{\gamma}]^3}{(1-\lambda^2/h^2)^{h^2}[1-k_1(1-\lambda^2/h^2)^{\gamma}(1-M^2)]}\right)^{1/2} \tag{19}$$

$$ds = \pm\left(\frac{(1-\lambda^2)[1-k_1(1-\lambda^2/h^2)^{\gamma}(1-M^2)]}{(1-\lambda^2/h^2)[1-k_1(1-\lambda^2/h^2)^{\gamma}]}\right)^{1/2}\frac{d\lambda}{\lambda}$$

$$\left(k_1 = \frac{k^2}{4\pi}\left(\frac{\varkappa+1}{2\varkappa}a_*^2\right)^{\gamma},\ h^2 = \frac{\varkappa+1}{\varkappa-1},\ \gamma = \frac{1}{\varkappa-1},\ \varkappa \neq 1\right)$$

where λ is the magnitude of the relative velocity and a_* is the critical velocity of sound. If $k_1 < 1$, the system (13) is correct for $\lambda < 1$ and the system (18) is correct for $\lambda > 1$. The case $k_1 > 1$ is of greater interest. Then the quantities

$$1 - k_1(1-M^2)(1-\lambda^2/h^2)^{\gamma}, \qquad 1 - k_1(1-\lambda^2/h^2)^{\gamma}$$

which are negative in the neighborhood of $\lambda = 0$ vanish respectively for $\lambda_1 < 1$ and $\lambda_2 = h(1 - k_1^{1-\kappa})$, where $\lambda_2 > \lambda_1$. In the interval of velocity variation $0 < \lambda < \lambda_1$ we have for every λ_2 the elliptic system of equations (13). If $\lambda_2 < 1$, we have for the subsonic interval $\lambda_1 < \lambda < \lambda_2$ the hyperbolic system of equations (18), and subsequently for the interval

$\lambda_2 < \lambda < 1$ the elliptic system of equations (13), and finally for $\lambda > 1$ again the system (18). If $\lambda_2 > 1$, the system (18) is correct for the intervals $\lambda_1 < \lambda < 1$ and $\lambda_2 < \lambda < h$, and the elliptic system of equations (13) for the supersonic interval $1 < \lambda < \lambda_2$. If

$$\lambda_2 = 1, \text{ или } k_1 = \left(\frac{h}{h-1}\right)^{\gamma}$$

the system (13) is correct for the whole interval $\lambda_1 < \lambda < h$.

Extracting the principal parts of the formulas (19) in the neighborhood of the singular points λ_1 and $\lambda_2 \neq 1$, we find that in the first case $\sqrt{K} \approx \text{const } s^{-1/3}$ and in the second case $\sqrt{K} = \text{const } s$, where s is to be-computed respectively for λ_1 and λ_2.

For $k = 0$ we have the usual equations of Chaplygin. For separate formulas $p = p(\rho)$ and values k_1 a system of equations in Legendre functions can be found which are more convenient for solution than the system (13).

We will pass from the functions ϕ, ψ to the functions Φ, Ψ by means of the Legendre transformation

$$\Phi = x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} - \varphi, \qquad \Psi = x\frac{\partial\psi}{\partial x} + y\frac{\partial\psi}{\partial y} - \psi \tag{20}$$

We have

$$x = \Phi_{u^*} = -\Psi_t, \qquad y = \Phi_{v^*} = \Psi_r \qquad \left(r = \frac{\cos\theta}{Q},\ t = \frac{\sin\theta}{Q}\right) \tag{21}$$

In the independent variables s, θ the system (21) has the following form (see, for instance, [7])

$$\frac{\partial\Psi}{\partial\theta} = -\sqrt{K_1}\,\frac{\partial\Phi}{\partial s}, \quad \frac{\partial\Psi}{\partial s} = \sqrt{K_1}\,\frac{\partial\Phi}{\partial\theta} \qquad \left(\sqrt{K_1} = \sqrt{K}\left[\frac{P}{Q}\right]^2\right)$$

Approximate and exact methods of solution of the Chaplygin equations can be used for the solution of problems of a given flow of gas in a magnetic field. For the approximations one must use a closure condition. For instance, if in the order of the approximation some other function $f(s)$ is taken instead of the rigorous dependence on $\sqrt{K(s)}$, after substituting the function $f(s)$ in place of $\sqrt{K}$ in Equation (14) we obtain an equation for determining the functions $P(s)$ and $Q(s)$. We obtain the dependence of ρ on w in the parametric form $\rho = \rho(s)$, $w = w(s)$ by means of the formulas $w^* = P^{-1}$, $\rho^* = P/Q$ and the formulas (8) and (9).

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